

# Exploitation of Sensitivity Derivatives for Improving Sampling Methods

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Many application codes, such as finite element structural analyses and computational fluid dynamics codes, are capable of producing many sensitivity derivatives at a small fraction of the cost of the underlying analysis. A simple variance reduction method is described that exploits such inexpensive sensitivity derivatives to increase the accuracy of sampling methods. Five examples, including a finite element structural analysis of an aircraft wing, are provided that illustrate an order of magnitude improvement in accuracy for both Monte Carlo and stratified sampling schemes.

## I. Introduction

SAMPLING methods for evaluating moments and distributions of random functions have been used extensively, but relatively little attention has been paid to utilizing sensitivity derivatives of the random function to improve the efficiency of sampling methods. (A sensitivity derivative is the derivative of the dependent random function with respect to one of the independent random variables.)

Recently, Cao et al.<sup>1</sup> (CHZ) formulated a sampling method for stochastic optimal control problems that exploits the sensitivity derivatives. There appear to have been no previous attempts in the engineering community to exploit derivative information in Monte Carlo methods for uncertainty analysis. For example, this possibility is not mentioned in the recent texts by Fishman<sup>2</sup> and Liu.<sup>3</sup> However, the mathematical finance community has recently developed related methods.<sup>4,5</sup>

A variety of engineering analyses are capable of producing sensitivity derivatives at a small fraction of the cost of the analysis itself. This is certainly true of many applications of finite element structural analysis. For example, data in Storaasli et al.<sup>6</sup> (Table 1, p. 350) indicate that for a relatively small finite element structural model [16,000 degrees of freedom (DOF)], a single derivative can be obtained in 7% of the time for an analysis. Because this relative time is inversely proportional to problem size, the relative cost of a derivative drops below 1% of the analysis time for 130,000 DOF. The recent development of efficient adjoint solvers for computational-fluid-dynamics (CFD) codes indicates that aerodynamic sensitivity derivatives can be obtained very efficiently. As one example, we cite the work of Carle et al.,<sup>7</sup> who reported that they have obtained 88 derivatives for Euler CFD at the cost of 10 analyses. As another, more dramatic example, we refer to the work of Sundaram et al.,<sup>8</sup> who have obtained 400 derivatives at the cost of 10 analyses for

viscous, turbulent CFD. Thus, there are important applications in which derivative information, even for tens of parameters, can be obtained at less cost than an analysis. The challenge is to devise sampling methods that exploit this additional inexpensive information to reduce the overall computational cost.

On their optimal control problem application, CHZ demonstrated that, compared with conventional Monte Carlo sampling, exploiting the sensitivity derivatives produced an order of magnitude increase in the efficiency of the sampling method on a model problem with one random variable. The present paper furnishes additional numerical support on and insight into the benefits of the use of sensitivity derivatives. In particular, we demonstrate 1) that this improved efficiency is even greater when the baseline sampling scheme is stratified sampling; 2) that improved efficiency is realized on a moderately complex, finite element analysis of an aircraft wing structure; and 3) that the improved efficiency extends to problems with more than one random variable. On the other hand, whereas CHZ's work was in the context of optimal control problems, our demonstrations are confined to simply the estimation of first and second moments of random functions.

The paper is organized as follows. In Sec. II, we summarize the relevant formulation of sensitivity derivative-enhanced sampling (SDES) methods from CHZ. In Sec. III we present the verification of our Monte Carlo and stratified sampling procedures on an analytical function of several variables. In Sec. IV, we demonstrate SDES using stratified sampling on the Burgers equation problem studied by CHZ, and in Sec. V we illustrate SDES on two standard test cases from the Society of Automotive Engineers. Finally, in Sec. V, we present results for an aircraft wing structure.

## II. Sensitivity Derivative-Enhanced Sampling Framework

Consider a real-valued function  $y(\xi)$ , where  $\xi$  is a real-valued random variable with probability density function  $\rho(\xi)$ . We assume that the sensitivity derivatives of  $y$  with respect to  $\xi$  are available. Let  $J(y)$  be a functional of  $y$ . The expected value of  $J(y)$ , denoted by  $E(J)$ , is given by

$$E(J) = \int J[y(\xi)]\rho(\xi) d\xi \quad (1)$$

Let  $V(J)$  denote its variance:

$$V(J) = \int \{J[y(\xi)] - E(J)\}^2 \rho(\xi) d\xi \quad (2)$$

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We use  $\bar{\xi}$  to denote the mean (or expected) value of  $\xi$ . The most straightforward way to compute the expected value of  $J$  is to use a Monte Carlo method. The problem with the Monte Carlo method is its slow convergence. It can easily take hundreds or thousands of samples to obtain satisfactory approximations to the moments.

In a Monte Carlo method, the approximation of the integral (1) is given by

$$\hat{J}^{MC} \approx \frac{1}{N} \sum_{i=1}^N J[y(\chi_i)] \quad (3)$$

where  $\chi_1, \chi_2, \dots, \chi_N$  is a sequence of samples of  $\xi$  generated according to the density function  $\rho(\xi)$ . The convergence of Eq. (3) is, of course, guaranteed by the large number theorem. But the approximation error in Eq. (3) is proportional to  $\sqrt{[V(J)/N]}$ . One naturally looks for ways to reduce variance to improve convergence. The current effort exploits the information regarding the sensitivity of the function  $J[y(\xi)]$  with respect to the stochastic parameter  $\xi$  to achieve variance reduction.

Let  $J_1(\xi)$  be the linear Taylor expansion of  $J$  at  $\bar{\xi}$ , that is,

$$J_1(\xi) = J[y(\bar{\xi})] + J_y[y(\bar{\xi})]y_\xi(\bar{\xi})(\xi - \bar{\xi}) \quad (4)$$

where  $y_\xi$  is the sensitivity of  $y$  with respect to  $\xi$ . Notice that

$$\int \{J[y(\xi)] - J_1(\xi)\} \rho(\xi) d\xi = \int J[y(\xi)] \rho(\xi) d\xi - J[y(\bar{\xi})]$$

This suggests the following sensitivity-derivative-enhanced Monte Carlo approximation of  $E(J)$ :

$$\hat{J}^{SDMC} \approx J[y(\bar{\xi})] + \frac{1}{N} \sum_{i=1}^N \{J[y(\chi_i)] - J_1(\chi_i)\} \quad (5)$$

We emphasize that we are not performing sampling on the approximation (4), but rather that our sampling actually uses the full analysis. Therefore our sampling procedure converges to the exact result for the original problem and not to the exact result of an approximation to the original problem (as in the popular resort to sampling response surface approximations rather than sampling the full analysis).

The variance of  $J[y(\xi)] - J_1(\xi)$  is given by

$$\int \{J[y(\xi)] - \overline{J[y(\xi)]} - J_y[y(\bar{\xi})] - y_\xi(\bar{\xi})(\xi - \bar{\xi})\}^2 \rho(\xi) d\xi \quad (6)$$

where  $\overline{J[y(\xi)]}$  is the mean of  $J[y(\xi)]$ . In the following theorem, we use the variance of  $\xi$  to estimate the variance of  $J - J_1$ . Without loss of generality, we assume that  $\xi$  is a scalar random variable. CHZ proved the following result, which is repeated here for completeness.

Let  $m = \max|(d/d\xi)J[y(\xi)]|$  and  $M = \max|(d^2/d\xi^2)J[y(\xi)]|$ . The following estimates hold:

$$V(J) \leq 2m^2 V(\xi) \quad (7)$$

$$V(J - J_1) \leq (M^2/2)\{V^2(\xi) + E[(\xi - \bar{\xi})^4]\} \quad (8)$$

*Proof:* The proof of the first inequality is straightforward. We only provide a proof for the second inequality. By the Taylor remainder formula there exists  $\xi_1$  such that

$$J[y(\phi)] - J[y(\bar{\xi})] = J_y[y(\bar{\xi})]y_\xi(\bar{\xi})(\phi - \bar{\xi})$$

$$+ \frac{1}{2} \frac{d^2}{d\xi^2} J[y(\xi)]|_{\xi=\xi_1} (\phi - \bar{\xi})^2$$

Because  $\bar{\xi}$  is the expectation of  $\xi$ , we have that

$$\overline{J[y(\xi)]} - J[y(\bar{\xi})] = \int \frac{1}{2} \frac{d^2}{d\xi^2} J[y(\xi)]|_{\xi=\xi_1} (\phi - \bar{\xi})^2 \rho(\phi) d\phi$$

Thus

$$|\overline{J[y(\xi)]} - J[y(\bar{\xi})]| \leq \frac{M}{2} \int (\phi - \bar{\xi})^2 \rho(\phi) d\phi = \frac{M}{2} V(\xi) \quad (9)$$

Using the Taylor remainder formula, we get

$$|J[y(\xi)] - J[y(\bar{\xi})] - J_y[y(\bar{\xi})]y_\xi(\bar{\xi})(\xi - \bar{\xi})| \leq (M/2)(\xi - \bar{\xi})^2$$

Combining Eqs. (6), (9), and the preceding inequality yields

$$\begin{aligned} V(J - J_1) &= \int \{J[y(\xi)] - J[y(\bar{\xi})] - J_y[y(\bar{\xi})]y_\xi(\bar{\xi})(\xi - \bar{\xi}) \\ &\quad - \{\overline{J[y(\xi)]} - J[y(\bar{\xi})] - J_y[y(\bar{\xi})]y_\xi(\bar{\xi})(\xi - \bar{\xi})\}^2 d\xi \\ &\leq 2 \int \{J[y(\xi)] - J[y(\bar{\xi})] - J_y[y(\bar{\xi})]y_\xi(\bar{\xi})(\xi - \bar{\xi})\}^2 \\ &\quad \times \rho(\xi) d\xi + 2 \int \{\overline{J[y(\xi)]} - J[y(\bar{\xi})]\}^2 \rho(\xi) d\xi \\ &\leq 2 \int \frac{M^2}{4} (\xi - \bar{\xi})^4 \rho(\xi) d\xi + \frac{M^2}{2} V^2(\xi) \\ &= \frac{M^2}{2} \{V^2(\xi) + E[(\xi - \bar{\xi})^4]\} \end{aligned}$$

This completes the proof. These results extend to functions of multiple random variables in obvious fashion.

The foregoing analysis indicates that the SDES method is effective when the variance of  $\xi$  is small. In the forthcoming examples, we focus on the first and second moments of  $y$ . In the former case,

$$J_1(\xi) = \bar{y} + y_\xi(\bar{\xi})(\xi - \bar{\xi})$$

and in the latter case

$$J_1(\xi) = \bar{y}^2 + 2\bar{y}y_\xi(\bar{\xi})(\xi - \bar{\xi})$$

### III. Verification and Evaluation of Sampling Procedures

Two different sampling procedures are considered in this work. One is the vanilla Monte Carlo method, given by Eq. (3). The other is stratified sampling, which we describe in the case of one random variable. (Because these are the first applications of this technique to problems with multiple random variables, we start with the classical stratified sampling method rather than the more sophisticated Hammersley stratified sampling and Latin hypercube sampling methods.) Let  $\Phi(\xi)$  denote the cumulative distribution function of the random variable  $\xi$ , that is,

$$\Phi(\xi) = \int_{-\infty}^{\xi} \rho(\zeta) d\zeta \quad (10)$$

The function  $\Phi$  is nondecreasing with range  $[0, 1]$ . The interval  $[0, 1]$  is divided into  $S$  strata, assumed here for simplicity to be of equal length:

$$[\eta^s, \eta^{s+1}], \quad s = 0, 1, \dots, S-1 \quad (11)$$

where

$$\eta^s = s/S, \quad s = 0, 1, \dots, S \quad (12)$$

In the standard stratified sampling method, for each  $s$  one chooses  $N_s$  random samples,  $\psi_i^s, i = 1, \dots, N_s$ , uniformly distributed in  $[\eta^s, \eta^{s+1}]$ , and computes the corresponding random samples in the variable  $\xi$  by inverting the cumulative distribution function:

$$\chi_i^s = \Phi^{-1}(\psi_i^s), \quad i = 1, \dots, N_s \quad (13)$$

This procedure assures that the  $\chi_i^s$  are distributed according to the density function  $\rho(\xi)$ . The expected value of  $J$  is then approximated by

$$\hat{J} \approx \frac{1}{SN_S} \sum_{s=0}^{S-1} \sum_{i=1}^{N_S} J[y(\chi_i^s)] \quad (14)$$

For the SDES version of stratified sampling, one first computes the contribution to Eq. (14) from each stratum by an application of Eq. (5). In particular,

$$\hat{J} \approx (1/S) \tilde{J}^s(y) \quad (15)$$

where

$$\tilde{J}^s(y) = J[y(\bar{\xi}^s)] + \frac{1}{N_S} \sum_{i=1}^{N_S} \{J[y(\chi_i^s)] - J_1^s(\chi_i^s)\} \quad (16)$$

with

$$J_1^s(\xi) = J[y(\bar{\xi}^s)] + J_y[y(\bar{\xi}^s)] y_{\xi}(\bar{\xi}^s)(\xi - \bar{\xi}^s) \quad (17)$$

where  $\bar{\xi}^s$  is the mean value of  $\xi$  in the  $s$ th stratum, given by

$$\bar{\xi}^s = \frac{\int_{\xi^s}^{\xi^{s+1}} \xi \rho(\xi) d\xi}{\int_{\xi^s}^{\xi^{s+1}} \rho(\xi) d\xi} \quad (18)$$

where  $\xi^s$  is computed from the  $\eta^s$  from Eq. (13). Note that the SDES method makes more use of sensitivity information for stratified sampling than for the Monte Carlo method, that is,  $S$  sensitivity derivatives are used in the former case and only one in the latter. For  $d$  random variables the SDES stratified sampling method uses  $S^d$  sets of sensitivity derivatives, whereas the SDES Monte Carlo method still just uses a single set of sensitivity derivatives. (A set of sensitivity derivatives consists of the  $d$  derivatives of  $y$  with respect to the  $d$  random variables.)

To verify our procedures and select performance measures for Monte Carlo sampling, stratified sampling, and their sensitivity derivative-enhanced variants, we have conducted tests on the simple function of  $d$  random variables:

$$y(\xi) = \sum_{j=1}^d \xi_j \quad (19)$$

We treat the  $d$  random variables as independent. In one test case their multidimensional density function is the Gaussian

$$\rho(\xi) = \prod_{j=1}^d \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(\xi_j - \bar{\xi}_j)^2}{2\sigma^2}\right] \quad (20)$$

Here,

$$\bar{y} = d, \quad \overline{y^2} = d^2 + d\sigma^2 \quad (21)$$

The numerator in the formula (18) for the mean values of  $\xi$  in each strata  $\bar{\xi}^s$  is evaluated analytically, and the denominator is computed with the aid of an IMSL routine for the cumulative distribution function for the standard normal distribution. We use  $\bar{\xi}_i = 1$  and  $\sigma_i = 0.10$  for all variables in this model problem example.

In the other test case the density function is the uniform distribution

$$\rho(\xi) = \begin{cases} 1 & \text{if } \xi_1, \dots, \xi_d \in [0.5, 1.5] \\ 0 & \text{otherwise} \end{cases} \quad (22)$$

for which

$$\bar{y} = d, \quad \overline{y^2} = d^2 + d/12 \quad (23)$$

Our focus in this paper is on quantifying to what extent the SDES method provides a better estimate than the baseline sampling schemes. There are many possible measures of this improvement. For  $N$  random samples the relative error of the Monte Carlo method (3) is  $\mathcal{E}^{MC} = [\hat{J}^{MC} - E(J)]/E(J)$ . Likewise, the relative error of the sensitivity derivative-enhanced Monte Carlo method (5) is  $\mathcal{E}^{SDMC} = [\hat{J}^{SDMC} - E(J)]/E(J)$ . These errors are themselves random variables with mean 0. An examination of the distribution of the individual results for 1000 sets of Monte Carlo estimates using  $N = 1024$  samples on the uniform distribution case, Fig. 1, is instructive. This figure shows the distribution of the ratio  $\mathcal{R}_j$  between the absolute values of the Monte Carlo (MC) error and the sensitivity derivative-enhanced Monte Carlo (SDMC) error for the  $j$ th set of  $N = 1024$  samples, that is,

$$\mathcal{R}_j = \frac{|\mathcal{E}_j^{MC}|}{|\mathcal{E}_j^{SDMC}|} \quad (24)$$

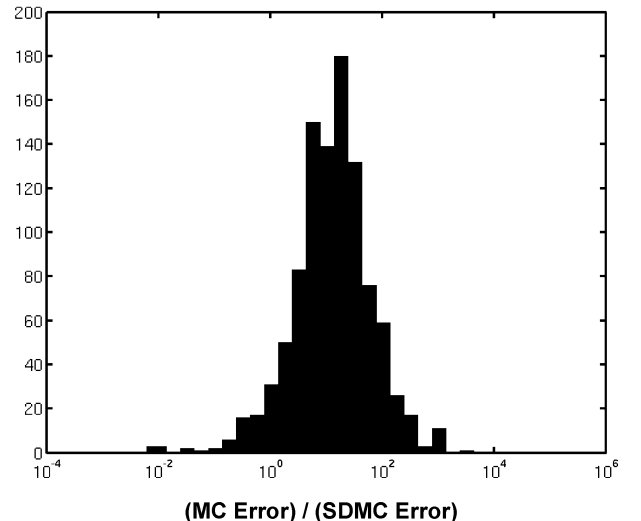
Because the errors  $\mathcal{E}_j^{MC}$  and  $\mathcal{E}_j^{SDMC}$  are themselves random variables (albeit with some correlation because they are computed from the same random sample), the ratio  $\mathcal{R}_j$  is also random. (Figure 1 indicates that the distribution of this ratio is closer to a log normal than to a normal distribution.) Clearly, in the vast majority of the cases the SDMC method produces a more accurate estimate than the MC method. But there is a small probability, in this case about 0.4%, that the SDMC result will produce a worse estimate.

Table 1 lists the results of various error measures for the Monte Carlo sampling on this problem as a function of  $N$ , using  $M = 1000$

**Table 1** Various measures for the improvement of the SDES method over regular MC sampling for the second moment of the linear test problem with a Gaussian distribution

$N$	Mean, $\mathcal{R}$	Median, $\mathcal{R}$	$\mathcal{I}_{L_1}$	$\mathcal{I}_{L_2}$
8	46.1	14.8	14.3	13.9
16	94.9	14.7	14.4	14.2
32	60.5	13.4	13.9	13.8
64	66.3	13.5	13.4	13.1
128	71.5	13.5	13.9	13.9
256	78.2	14.2	14.0	14.0
512	192	13.9	13.7	13.5
1,024	47.8	14.0	13.3	13.3
2,048	94.4	14.3	14.2	14.1
4,096	78.1	13.8	13.5	13.4
8,192	54.7	15.6	14.8	14.8
16,384	86.9	14.2	14.2	13.8

**Frequency**



**Fig. 1** Distribution of the ratio of Monte Carlo sampling error to the SDES Monte Carlo sampling error on the linear test problem with a Gaussian distribution.

different sets of random samples. This table shows the mean and median values of  $\mathcal{R}_j$  along with the ratios of the  $L_1$  and  $L_2$  values of the errors, where these ratios are given by

$$\mathcal{I}_{L_1} = \mathcal{E}_{L_1}^{MC} / \mathcal{E}_{L_1}^{SDMC} \quad (25)$$

$$\mathcal{I}_{L_2} = \mathcal{E}_{L_2}^{MC} / \mathcal{E}_{L_2}^{SDMC} \quad (26)$$

where

$$\mathcal{E}_{L_1}^{MC} = \frac{1}{M} \sum_{j=1}^M |\mathcal{E}_j^{MC}| \quad (27)$$

$$\mathcal{E}_{L_2}^{MC} = \sqrt{\frac{1}{M} \sum_{j=1}^M [\mathcal{E}_j^{MC}]^2} \quad (28)$$

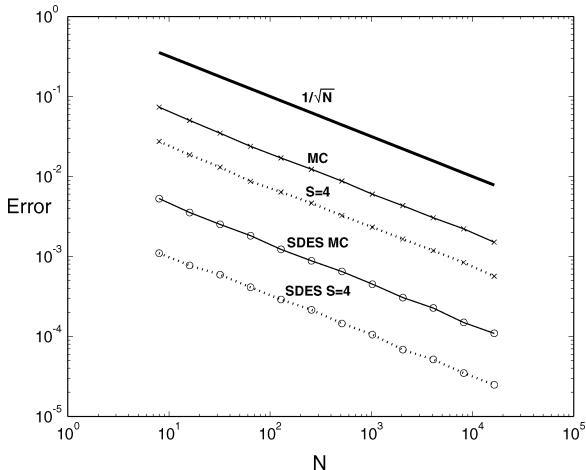
The mean value of  $\mathcal{R}$  has the largest value and the most variability of these measures, driven by the large influence of random fluctuations in the long tail illustrated in Fig. 1. The other three measures produce nearly identical results. Henceforth, we will report just the ratio  $\mathcal{I}_{L_2}$  of the  $L_2$  (or rms) relative errors. (The rms error is often used in the rigorous numerical analysis of sampling methods.<sup>9,10</sup>) We shall refer to this quantity as the SDES improvement ratio (SIR).

As expected for this linear test function, the SDES results for the estimates of the first moments  $\bar{y}$  are exact. Indeed, the numerical results for all test cases demonstrate this to the full 64-bit precision of the computations. (All computations in this paper were performed in 64-bit arithmetic.)

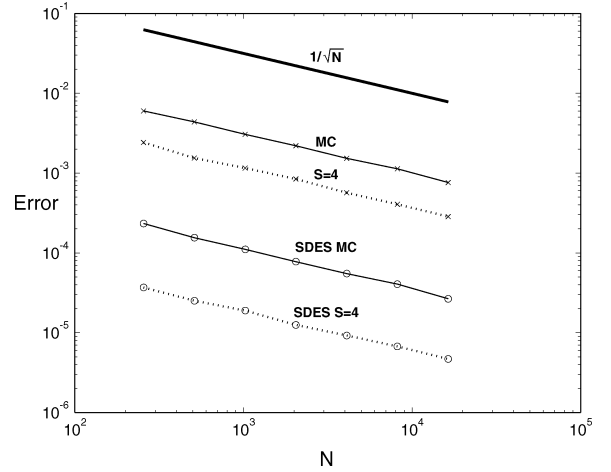
The effectiveness of the SDES approach on second moment estimates for the linear model problem with both Gaussian and uniform distributions is documented in Figs. 2–5 and Table 2. The figures illustrate the rms relative errors for MC sampling and stratified sampling with four strata ( $S = 4$ ), along with their SDES versions on estimates of the second moment  $y^2$  for a variety of sample sizes using  $M = 1000$  sets of samples. Results for both Gaussian and uniform distributions are reported, as are results for one random variable and for four random variables. (Note that for the stratified sampling cases, the number of samples in each stratum is  $N_S = N/S^d$ ). The exact values for  $E(J)$  are taken from Eqs. (21) and (23). In all cases

**Table 2** SDES improvement ratios for second-moment estimates of the model problem using  $N = 1024$  and  $M = 1000$

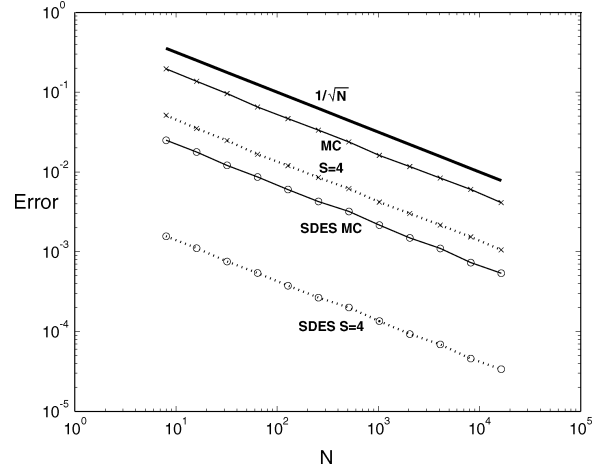
Distribution	1 variable		4 variables	
	MC ratio	$S = 4$ ratio	MC ratio	$S = 4$ ratio
Gaussian	13	22	28	61
Uniform	7	31	11	41



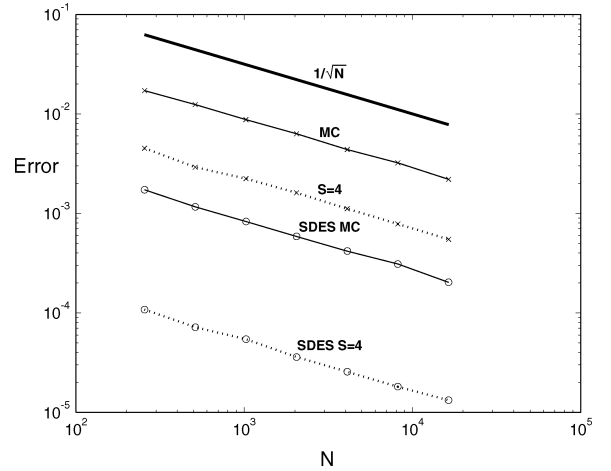
**Fig. 2** Relative errors in the second-moment estimates for the model problem for one random variable with a Gaussian distribution.



**Fig. 3** Relative errors in the second-moment estimates for the model problem for four random variables with Gaussian distributions.



**Fig. 4** Relative errors in the second-moment estimates for the model problem for one random variable with a uniform distribution.



**Fig. 5** Relative errors in the second-moment estimates for the model problem for four random variables with uniform distributions.

the various sampling methods exhibit the expected  $1/\sqrt{N}$  error decay.

Table 2 provides the SDES improvement ratios (SIRs) for the  $N = 1024$  cases shown in the figures. The results here and in the four figures indicate that 1) the estimates provided by the SDES method are typically one to two orders of magnitude more accurate than those of the underlying sampling scheme, 2) the relative improvement is greater for the more sophisticated stratified sampling

scheme than for the basic Monte Carlo method, and 3) the relative improvement is greater for the larger number of random variables. Furthermore, note that except for the four-variable,  $S = 4$  case, the cost of the sensitivity derivatives is a negligible component of the total cost.

#### IV. Demonstration on a Solution of Burgers Equation

CHZ's main example was based on the generalized steady-state Burgers equation<sup>11</sup>

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} \left( v \frac{\partial y}{\partial x} \right) \quad \text{for} \quad x \in \left( -\frac{1}{2}, \frac{1}{2} \right) \\ f(y) &= \frac{1}{2} y(1 - y) \\ y\left(-\frac{1}{2}\right) &= \frac{1}{2} \left[ 1 + \tanh\left(\frac{-1}{8v}\right) \right] \\ y\left(\frac{1}{2}\right) &= \frac{1}{2} \left[ 1 + \tanh\left(\frac{1}{8v}\right) \right] \end{aligned} \quad (29)$$

This equation has the exact solution

$$y(x; v) = \frac{1}{2} [1 + \tanh(x/4v)] \quad (30)$$

The parameter  $v$  (viscosity) is treated as a random variable. As a result, the solution  $y = y(x; v)$  of Burgers equation is also a random function. Whereas CHZ considered this problem in the context of optimal control, here we confine ourselves to just the estimation of the first and second moments.

We again consider both a Gaussian distribution and a uniform distribution. To conform to the cases from CHZ, the parameters of this Gaussian are  $\bar{v} = 2.0$  and  $\sigma = 0.1$ , with the Gaussian cutoff below  $v = 0.1$  and above  $v = 3.9$ . For this reason we use a quadrature formula (Simpson's rule with 100 intervals) to evaluate the integrals in Eq. (18). The uniform distribution case has the probability density function

$$\rho(\xi) = \begin{cases} 1 & \text{if } \xi \in [0.1, 0.3] \\ 0 & \text{otherwise} \end{cases} \quad (31)$$

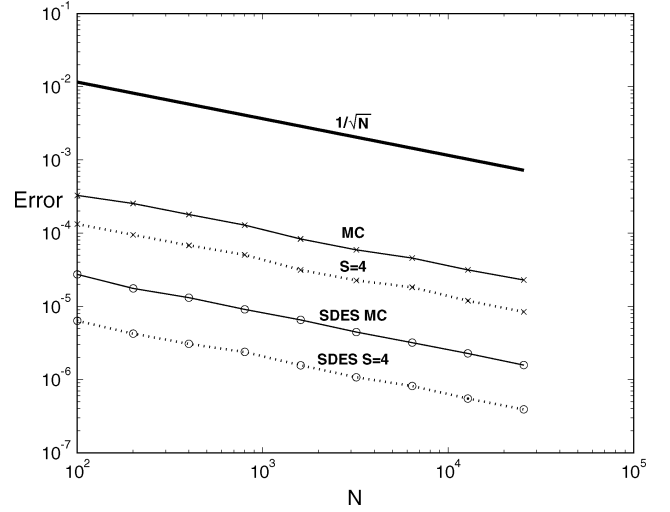
We revisit the example here to illustrate the improvement that the SDES method provides over the baseline stratified sampling scheme. We include conventional MC for reference. Our figure of merit is the rms error (in  $x$ ) of the approximation of the second moment of  $y(x; v)$ :

$$E[y^2(x)] = \int y^2(x; v) \rho(v) dv$$

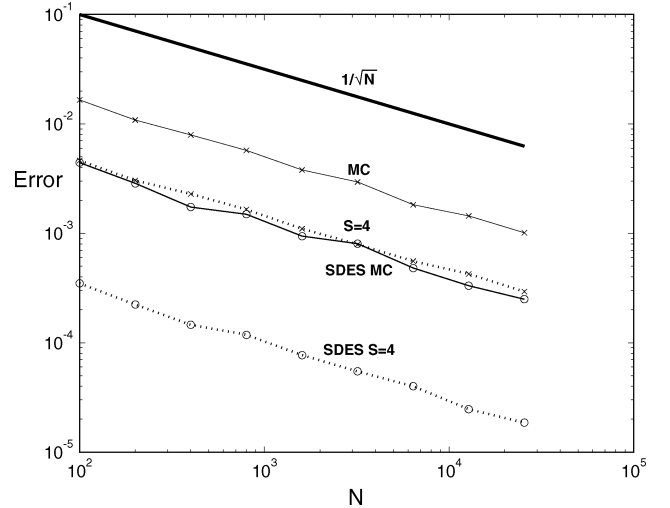
The exact value of the second moment, as well as the rms errors, are evaluated by numerical quadrature. (In our numerical examples, we use Simpson's rule with 100 intervals to compute the exact values of the second moments at each  $x$  and 40 intervals to compute the rms errors over  $x$ .) Figures 6 and 7 illustrate the relative errors as a function of  $N$  for Monte Carlo and for stratified sampling with  $S = 4$ , both with and without the SDES procedure. For these examples we averaged over only  $M = 20$  sets of samples because the integration already reduced the statistical fluctuations. The results indicate that 1) the errors of all methods decay at the expected  $1/\sqrt{N}$  rate; 2) the stratified sampling methods afford a significant improvement over conventional Monte Carlo; and 3) the SDES approach achieves an order of magnitude reduction in the error. Table 3 documents further the final point. It reports the SIRs, including cases of stratified sampling with  $S = 8$ . The corresponding results for the rms error in the first moments (not shown here) give ratios that are about 10% greater than those for the second moment.

**Table 3** SDES improvement ratios for second-moment estimates of the Burgers problem

Distribution	MC ratio	$S = 4$ ratio	$S = 8$ ratio
Gaussian	12.4	21.1	22.5
Uniform	3.5	13.4	25.8



**Fig. 6** Relative errors in the second-moment estimates for the Burgers problem with a Gaussian distribution.



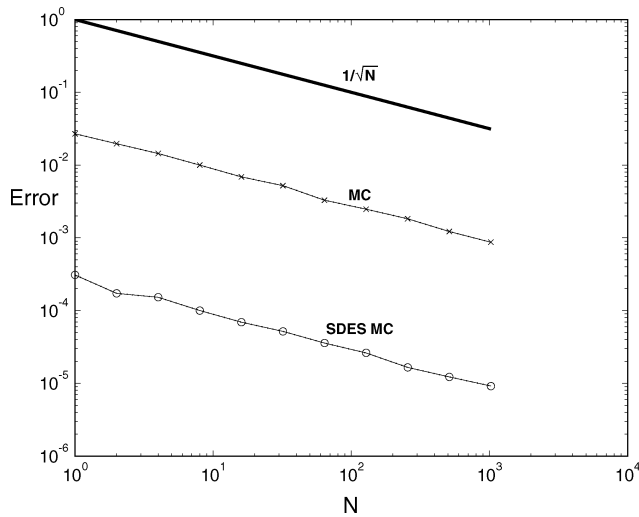
**Fig. 7** Relative errors in the second-moment estimates for the Burgers problem with a uniform distribution.

#### V. Two Society of Automotive Engineers Test Cases

The Probabilistics Methods Committee of the Society of Automotive Engineers (SAE) has selected 11 standard test cases for probabilistic methods. Here we demonstrate the SDES method on two of these. In SAE case 2a, the function is the same as the first test case in this paper, but with 50 random variables. The random variables all have mean  $\bar{\xi}_i = 1$  and standard deviation  $\sigma_i = 0.10$ , just as in our preceding example. However, because the SDES result is exact for the first moment we report the results for the second moment. Table 4 lists the rms relative errors computed with  $M = 1000$  sets of samples of the MC and SDES simulations for the second moment along with the SDES improvement factors. (Stratified sampling is infeasible for this case, as there would need to be  $S^{50}$  strata.) As we observed for the preceding results for this linear problem with one and four variables, the SDES improvement factor increases with the number of random variables. Figure 8 indicates the expected  $1/\sqrt{N}$  error decay for both the MC and SDES simulations.

**Table 4** Relative errors and SDES improvement ratios for SAE case 2a

$N$	MC error	SDES error	Error ratio
1	2.6875e-2	3.0866e-4	99.21
2	1.9616e-2	1.7233e-4	98.35
4	1.4413e-2	1.5221e-4	98.66
8	9.9740e-3	1.0026e-4	104.04
16	6.8826e-3	6.9306e-5	100.27
32	5.2030e-3	5.1790e-5	93.08
64	3.2897e-3	3.5829e-5	95.45
128	2.4715e-3	2.6167e-5	100.43
256	1.8232e-3	1.6549e-5	101.82
512	1.2218e-3	1.2238e-5	95.64
1024	8.6903e-4	9.1982e-6	92.47
2048	6.2228e-4	6.8266e-6	91.16
4096	4.5351e-4	4.8670e-6	93.21

**Fig. 8** Relative errors in the second-moment estimates for SAE case 2a.

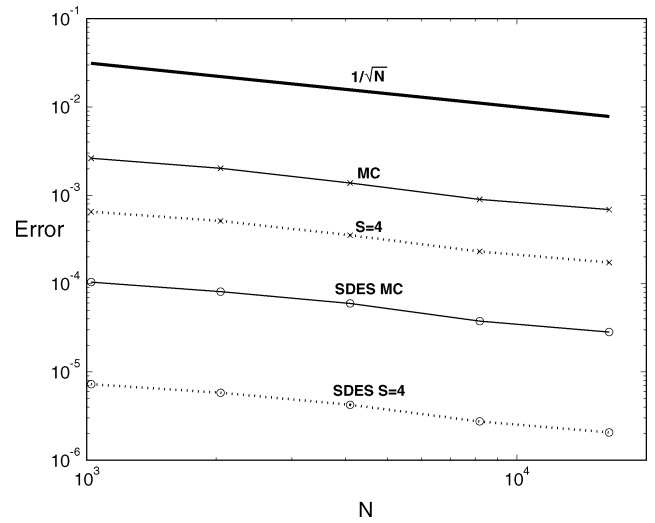
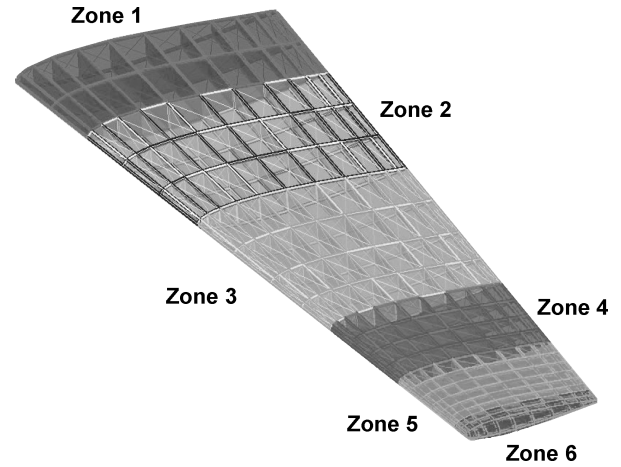
In SAE case 6, the response function is the maximum radial stress of a rotating disk computed from the equation

$$(\sigma_r)_{\max} = [(3 + \nu)/8][\rho/[(9.81)(39.37)]][\omega(2\pi/60)]^2(r_0^2 - r_i^2) \quad (32)$$

Here  $\nu$ ,  $\rho$ ,  $\omega$ ,  $r_0$ , and  $r_i$  are random variables with the following distributions:  $\nu$  = Poisson's ratio  $N(\mu = 0.30, \sigma = 0.005)$ ,  $\rho$  = density (lb/in.<sup>3</sup>)  $N(\mu = 0.284, \sigma = 0.002)$ ,  $\omega$  = rotor speed (rpm)  $U(10,000, 11,000)$ ,  $r_0$  = outer radius (in.)  $N(\mu = 8, \sigma = 0.02)$ , and  $r_i$  = inner radius (in.)  $N(\mu = 2, \sigma = 0.01)$ . The results using  $N = 1024$  with  $M = 1000$  sets of samples for both the first and second moments are given here. For the first moment, the MC SIR is 68, and the  $S = 4$  SIR is 212. For the second moment, the MC SIR is 25, and the  $S = 4$  SIR is 85. The exact values for the first and second moments are 22,020.4032 and 486,401,510.95934, respectively. For this test problem, the SDES approach provides several orders of magnitude improvement in sampling efficiency. Figure 9 once again indicates the expected  $1/\sqrt{N}$  error decay for all the simulations and the significant improvement of SDES over MC.

## VI. Aircraft Wing Structure Application

The final numerical example is for a structural analysis problem using finite element analysis. The specific problem is taken from the work of Gumbert et al.<sup>12</sup> (GHN). The trapezoidal-planform, semispan wing is illustrated in Fig. 10. The wing is divided into six zones, marked by the different shadings in the figure, with zone 1 near the wing root. The airfoil sections vary linearly from a NACA 0012 section at the root to a NACA 0008 section at the tip. The finite

**Fig. 9** Relative errors in the second-moment estimates for SAE case 6.**Fig. 10** Aircraft wing structural model.

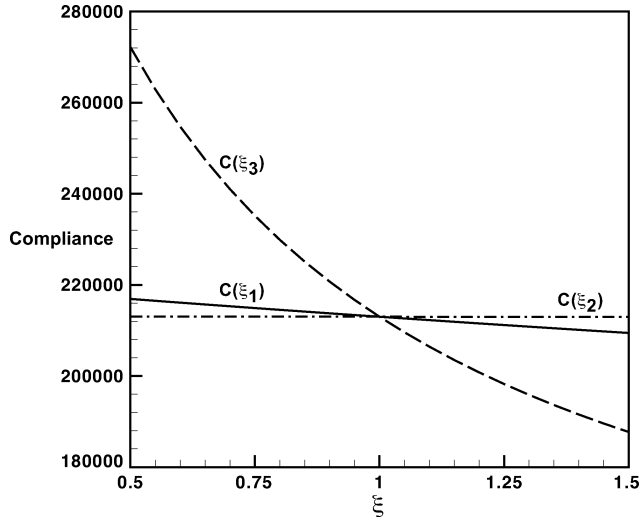
element model consists of 583 nodes, with 2141 constant-strain triangle (CST) elements and 1110 truss elements. Linear elasticity is assumed. We adopt the same grouping of structural thicknesses as used by GHN. In the case of the CST elements, which are all that are considered in the present work, there are three parameters for each zone. Whereas, for GHN, these parameters were design variables, for us they are the random variables. These parameters are multiplicative factors for the baseline values of the element thicknesses. For example, variable 1 is the thickness multiple for the skin elements in zone 1, variable 2 is the thickness multiple for the web elements of the ribs in zone 1, and variable 3 is the thickness multiple for the web elements of the spars in zone 1, variable 4 is the thickness multiple for the skin elements in zone 2, etc. These scaling parameters are denoted by  $\xi = (\xi_1, \dots, \xi_d)$ , where  $d$  denotes the number of parameters used in the particular case. The output functional of the analysis  $y(\xi)$  is the compliance, which is the work done by the aerodynamic pressure to deflect the structure. It is given by the integral over the wing of the aerodynamic pressures times the structural displacements. The wing leading edge has a 9.46-deg sweep, a root of 20 ft, and a span of 60 ft. The trailing edge is unswept. The pressures were based on a static aeroelastic computation (using Euler CFD and the finite element structural analysis) of the flow past the baseline wing at a freestream Mach number of 0.80 and an angle of attack of 1 deg. The baseline values of the first three parameters are 0.188, 0.0375, and 0.1200 in. The compliance is 213,285 lb-in. The finite element code is described by Hou et al.<sup>13</sup> It was developed under contract to NASA Langley Research Center to enable basic research on simultaneous, coupled aerostructural optimization using first-order sensitivity derivatives of both sizing and shape variables (see GHN

**Table 5** SDES improvement ratios for first- and second-moment estimates of the aircraft wing structure problem with one random variable

Variable	First moment		Second moment	
	MC	$S = 4$	MC	$S = 4$
1	42	160	35	140
2	8.0	25	8.0	25
3	4.3	14	3.5	11
4	44	170	38	150
5	10	36	10	36
6	4.3	14	3.5	11

**Table 6** SDES improvement ratios for first- and second-moment estimates of the aircraft wing structure problem with three random variables

Variables	First moment		Second moment	
	MC	$S = 4$	MC	$S = 4$
1,4,7	49	160	40	140
2,5,8	9.3	32	9.3	32
3,6,9	4.6	15	3.4	12
1,2,3	3.9	17	3.2	13

**Fig. 11** Compliance variation.

and their earlier papers for the optimization applications.) Gumbert et al.<sup>14</sup> have recently used this code for uncertainty analysis applications as well. A single structural analysis for this problem takes roughly 1/5 s of CPU time.

Numerical examples for this problem are given for a uniform distribution of the scaling parameters  $\xi_i$  on  $[0.5, 1.5]$ . The exact solution to the first and second moments is computed by numerical quadrature. The estimated accuracy is at least 10 digits for the one random variable cases and at least eight digits for the three random variable cases. Table 5 gives the results for the SDES improvement ratios for the first six structural variables (zones 1 and 2). The computations were made for  $N = 256$ . Because the cost of the function evaluation of this structures problem is much higher than for the earlier examples, we only compute 100 sets of samples, that is,  $M = 100$ .

The sensitivities of the compliance with respect to the first three thicknesses at their mean values are  $-396,777$ ,  $-1040$  and  $-591,496$  lb-in./in., respectively. The variation of the output (compliance) with respect to the first three random (scaling) variables is illustrated in Fig. 11. The percentage variations in the compliance with respect to the first three variables are 3.5, 0.02, and 40%, respectively. The relative impact of these variables upon the variation of the compliance is readily understandable on physical grounds. The effect of the aerodynamic load is felt primarily by the spars (variable 3), secondarily by the skin (variable 1), and hardly at all by the ribs (variable 2). The relatively small values of the SDES improvement factors for the rib variables are as a result of the relatively high accuracy of the Monte Carlo method when the output function has little variation. Indeed the relative error for the Monte Carlo approximation for this case is already of order  $10^{-5}$  for  $N = 8$ , whereas it is of order  $10^{-2}$  for the first variable and of order  $10^{-1}$  for the third variable. The compliance depends nearly linearly upon the first variable (skin thickness) and the second variable (rib web thickness), but has substantial curvature for the third variable (spar web thickness). Hence, we should expect the greater improvement

produced by the SDES method for the first variable than for the third variable.

Table 6 illustrates the sampling results for cases with three random variables. The first three rows of results indicate that the improvement ratios for the multivariable cases are comparable to those for the single-variable case, provided that the variables all behave similarly. In contrast, the fourth row indicates that when the variables have dramatically different impacts upon the output, then the multivariable results are comparable to those of the worst of the individual variable results.

Some computations have also been made for a Gaussian distribution of the random scaling variables (unit mean and standard deviation of 0.20). The results are roughly the same as for the uniform distribution.

We should note that there has been some promising work on obtaining second-order sensitivity derivatives efficiently from finite element structural codes<sup>13</sup> and CFD codes.<sup>15,16</sup>

## VII. Conclusions

In conclusion, we have furnished numerical results attesting to the advantage of exploiting sensitivity derivatives in sampling schemes. The examples range from analytic model problems to full finite element structural analyses. For a fixed number of samples, there is typically an order of magnitude reduction in the error achieved by the sensitivity derivative-enhanced sampling (SDES) approach. Equivalently, SDES computations require two orders of magnitude fewer samples to achieve the same accuracy in the moments compared with baseline Monte Carlo and stratified sampling schemes. The improvement afforded by the SDES approach appears to increase as the number of random variables increases. The overhead for the extra sensitivity derivative calculations is less than 5%, even for the structural analysis example.

The next steps in this project include extending the SDES method to exploit the semi-analytic second-order sensitivity derivatives available from some codes, evaluating the impact of SDES on Latin hypercube sampling in order to handle more than a handful of random variables and extending the application to the computation of cumulative distribution functions.

In the present work, we have only made very minor use of derivative information and have obtained a significant speed-up over two conventional sampling methods. This suggests that more attention should be devoted to exploiting relatively inexpensive sensitivity derivatives in traditional sampling methods. The long-term research challenge is to make even better use of derivative information in otherwise conventional sampling methods.

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